Extra Credit

Let

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{xy}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Let $R = [0,2] \times [0,1]$

Show

$$\int_0^2 \left(\int_0^1 f(x,y)\,dy \right) dx \neq \int_0^1 \left(\int_0^2 f(x,y)\,dx \right) dy$$

Is Fubini's Theorem wrong?

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$= \int_{-\infty}^{0} e^{-x^2}\,dx + \int_{0}^{\infty} e^{-x^2}\,dx$$

Now $\displaystyle\int_{0}^{\infty} e^{-x^2}\,dx = \lim_{t\to\infty} \int_{0}^{t} e^{-x^2}\,dx$

by definition, and since $f(x) = e^{-x^2}$

is even $(f(-x) = f(x))$,

$$\int_0^t e^{-x^2}\,dx = \int_{-t}^0 e^{-x^2}\,dx.$$

Then

$$\int_{-t}^t e^{-x^2}\,dx = \int_0^t e^{-x^2}\,dx + \int_{-t}^0 e^{-x^2}\,dx$$

$$= 2\int_0^t e^{-x^2}\,dx.$$

We will calculate

$$\left(2\int_0^t e^{-x^2}\,dx\right)^2$$

$$=\left(\int_{-t}^{t} e^{-x^2}\,dx\right)^2$$

$$=\int_{-t}^{t} e^{-x^2}\,dx\cdot\int_{-t}^{t} e^{-y^2}\,dy$$

$$= \int_R e^{-x^2-y^2} \, dA$$

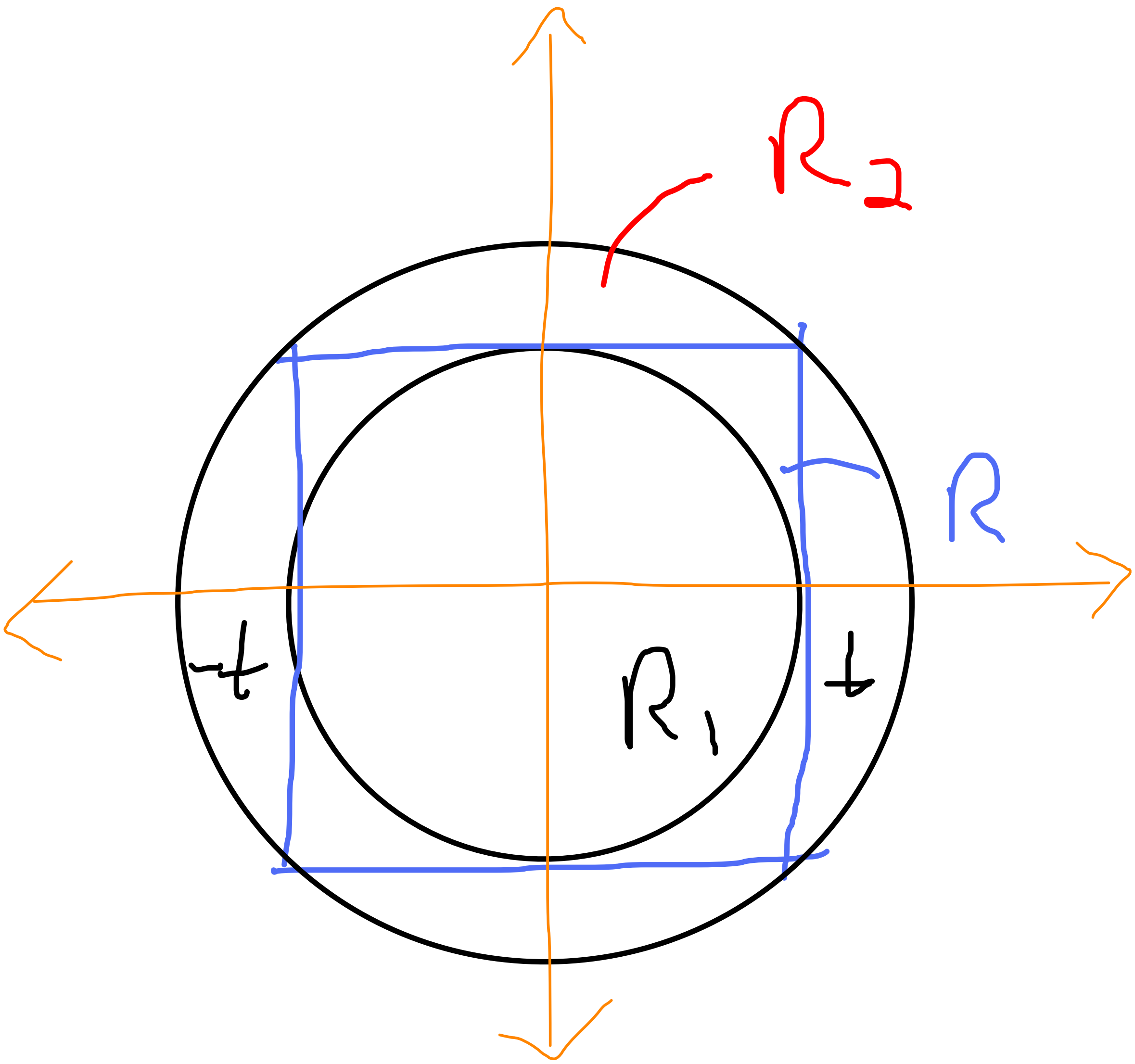
where $R = [-t, t] \times [-t, t]$.

Let $R_1 = \{(x,y): x^2+y^2 \leq t^2\}$

and

$R_2 = \{(x,y): x^2+y^2 \leq 2t^2\}$

Picture:



So R_1 is inside R and R is inside R_2

Then

$$\iint_{R_1} e^{-x^2-y^2}\,dA \le \iint_{R} e^{-x^2-y^2}\,dA$$

$$\le \iint_{R_2} e^{-x^2-y^2}\,dA$$

Since $e^{-x^2-y^2} \geq 0$ for all (x,y).

Now in polar coordinates,

$$R_1 = \{(r,\theta) : 0 \le \theta \le 2\pi,\ 0 \le r \le t\}$$

$$R_2 = \{(r,\theta) : 0 \le \theta \le 2\pi,\ 0 \le r \le \sqrt{2}t\}$$

so

$$\int_{R_1} e^{-x^2-y^2}\,dA = \int_0^{2\pi}\left(\int_0^{t} e^{-r^2} r\,dr\right)d\theta$$

and

$$\int_{R_2} e^{-x^2-y^2}\,dA = \int_0^{2\pi}\left(\int_0^{\sqrt{2}t} e^{-r^2} r\,dr\right)d\theta$$

Then $\displaystyle\iint_{R_1} e^{-x^2-y^2}\,dA$

$$= 2\pi \int_0^t e^{-r^2}\,r\,dr \qquad u = r^2,\ du = 2r\,dr$$

$$= \pi \int_0^{t^2} e^{-u}\,du = \pi\left(-e^{-u}\Big|_0^{t^2}\right)$$

$$= \pi\left(1 - e^{-t^2}\right)$$

and

$$\int_{R_2} e^{-x^2-y^2}\, dA$$

$$= 2\pi \int_0^{\sqrt{2}t} e^{-r^2}\, r\, dr \qquad u = r^2$$

$$du = 2r\,dr$$

$$= \pi \int_0^{2t^2} e^{-u}\, du = \pi\left(-e^{-u}\Big|_0^{2t^2}\right)$$

$$= \pi\left(1 - e^{-2t^2}\right)$$

Then

$$\pi(1-e^{-t^2}) \le \iint_R e^{-x^2-y^2}\,dA \le \pi(1-e^{-2t^2})$$

Remember $$\iint_R e^{-x^2-y^2}\,dA = \left(2\int_0^t e^{-x^2}\,dx\right)^2,$$

So

$$\boxed{\pi(1-e^{-t^2}) \le \left(2\int_0^t e^{-x^2}\,dx\right)^2 \le \pi(1-e^{-2t^2})}$$

Now take limit as $t \to \infty$.

$$\lim_{t \to \infty} e^{-t^2} = \lim_{t \to \infty} e^{-2t^2} = 0, \text{ so}$$

$$\pi \leq \lim_{t \to \infty} \left(2 \int_0^t e^{-x^2} dx \right)^2 \leq \pi$$

Then by the squeeze theorem,

$$2 \int_0^\infty e^{-x^2} dx = \lim_{t \to \infty} 2 \int_0^t e^{-x^2} dx = \sqrt{\pi}$$

Now since $\int_0^t e^{-x^2}\,dx = \int_{-t}^0 e^{-x^2}\,dx$,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_0^{\infty} e^{-x^2}\,dx,$$

So $\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$